

On the Completeness of the Quasinormal Modes of the Pöschl-Teller Potential

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The completeness of the quasinormal modes of the wave equation with Pöschl-Teller potential is investigated. A main result is that after a large enough time t_0 , the solutions of this equation corresponding to C^∞ -data with compact support can be expanded uniformly in time with respect to the quasinormal modes, thereby leading to absolutely convergent series. Explicit estimates for t_0 depending on both the support of the data and the point of observation are given. For the particular case of an “early” time and zero distance between the support of the data and observational point, it is shown that the corresponding series is *not* absolutely convergent, and hence that there is no associated sum which is independent of the order of summation.

I. GENERAL INTRODUCTION

The description of a compact classical system often leads to the consideration of a “small” perturbation of some special solution of its evolution equations. Expanding around this solution leads to a linear evolution equation for some perturbed quantities which characterise the system. For a system with no explicit time dependence, a further step consists of finding the *normal mode solutions* of the evolution equation satisfying certain physical boundary conditions. To provide a complete description of the system under small perturbations, every solution of the linear equation satisfying the boundary conditions must have an expansion in terms of these modes.

To my knowledge the only well-developed mathematical framework to date for deciding such a “completeness” question is provided by the spectral theory of linear operators in Hilbert spaces. This is the approach taken in this paper.

It is frequently the case (as in this paper) that the linear equation is a wave equation. Then, it is well-known that the *squares* of the normal modes frequencies coincide with the spectrum of that linear self-adjoint operator which is naturally connected with the equation and the boundary conditions. Since this spectrum is real, the normal mode frequencies are purely imaginary or real. Using the so called *functional calculus* associated with the operator, a *representation in terms of the normal modes* can be given for the solution of the initial-value problem for the linear equation.

Quasinormal mode solutions of the linear equation are often displayed by, in some sense, *dissipative* systems. They satisfy boundary conditions which differ from that for the normal modes, but usually are viewed in the same context as the normal mode solutions. From this point of view it is natural to ask whether they are in any sense complete [9]. On the other hand *quasinormal frequencies* have, in general, both real and imaginary parts and hence their *squares cannot belong to a spectrum of any linear self-adjoint operator*.

In the special case considered in this paper the system is initially contained in some finite box in space and is “*dissipative*”, if one considers the energy contained in the box as a function of time. But the system is *conservative* if one considers the energy distributed in the whole space. It turns out that the quasinormal frequencies of the “finite” system are *resonances* of the operator corresponding to the “infinite” system. The analogous can be seen to be true for many other systems.¹

This paper addresses the completeness question of the resonance modes of the infinite system using the framework of “spectral theory”. The system is described by a wave equation in one-dimensional space (as motivated by astrophysical systems). That the system is initially contained in a finite box is displayed by the fact that only initial values with compact support are considered.

¹Such resonances are known to be important in quantum theory and mathematical methods have been developed to deal with them (Volumes III and IV of [10]). However it is also known that the concept of resonances of an operator is far more delicate than that of the spectrum. In contrast to the spectrum, resonances depend not only on the operator, but also on the choice of dense subspace of the underlying function space. In addition much less is known about resonances than about spectra, concerning in particular their behaviour under perturbations of the operator.

II. INTRODUCTION

The decay in time of the solutions of the Einstein field equations linearized around the Schwarzschild metric is governed by quasinormal frequencies (“QNF”) and the corresponding modes (“QNM”) [12]. For perturbing fields of the form

$$\Phi(t, x, \theta, \varphi) := \frac{1}{r} \phi(t, x) \cdot \mathbf{Y}_{\ell m}(\theta, \varphi), \quad (1)$$

where $\mathbf{Y}_{\ell m}$ denotes an appropriate tensor spherical harmonic function; t, r, θ, φ are the usual Schwarzschild coordinate functions; $x := r + \ln(r - 1)$ is the “tortoise” coordinate function and ℓ is a natural number, one gets the following wave equation for the scalar function ϕ ,²

$$\frac{\partial^2 \phi}{\partial t^2} + \left(-\frac{\partial^2}{\partial x^2} + U \right) \phi = 0, \quad (2)$$

where

$$U(r) := \left(1 - \frac{1}{r} \right) \cdot \left(\frac{l(l+1)}{r^2} - \frac{3}{r^3} \right). \quad (3)$$

A still open mathematical question [9], is whether, and then in which sense, the solutions of (2) corresponding to C^∞ -data with compact support can be represented as sums of quasinormal mode solutions of (2). The latter are separated solutions satisfying so called “purely outgoing” boundary conditions (see e.g. [5]). Since there are an infinite number of such modes [2] it is in particular important to find out the type of convergence with respect to which such an expansion may be valid.

The answer to these questions is obscured by technical problems — the QNF are not explicitly known and there is no convenient analytical representation for the QNM.

In such a situation it is natural to ask whether there is any reason to expect that such a quasinormal mode expansion exists? Or more precisely, is there a wave equation of type (2) having infinitely many quasinormal modes such that each solution corresponding to C^∞ -data with compact support has an expansion into quasinormal modes? To my knowledge such a wave equation is not known. Hence it is still unclear whether one should expect such a “quasinormal mode expansion” for (2) to exist. Further, if such a normal mode expansion does exist for (2) a natural next step would be to ask whether this is true also for other wave equations, or in other words, whether the phenomenon is in any sense “stable” against “small perturbations” of the potential. Such points suggest the consideration of other wave equations than (2) and in this paper we now look at the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} + \left(-\frac{\partial^2}{\partial x^2} + V \right) \phi = 0, \quad (4)$$

where the potential V is the Pöschl-Teller potential [8],

$$V(x) := \frac{V_0}{\cosh^2(x/b)}, \quad x \in \mathbb{R}. \quad (5)$$

Here V_0 and b are, respectively, the maximal value and the “width” of V and are non zero positive real numbers (considered as given in the following). There are good reasons for working with this special choice of the so called “Pöschl-Teller” potential V instead of the Schwarzschild potential U . First, the QNF and QNM are known analytically [5], and there are an infinite number of QNF which are elementary functions of V_0 and b . In addition the shapes of U and V are similar (see Figure 1) and both potentials are integrable over the real line and decay exponentially for $x \rightarrow -\infty$. However, the decay of U and V differs for $x \rightarrow \infty$, where U decays as $1/r^2$ and V decays exponentially. These similarities have already been used in order to approximate the QNF of the Schwarzschild black hole which have “low” imaginary part by the corresponding QNF for V [5]. A final very important reason for considering this particular wave equation is that the resolvent of the Sturm-Liouville operator corresponding to V (given later in Equation (10)) can be given explicitly in terms of well-known analytic special functions. This cannot be done, so far,

²Here the units are chosen such that the Schwarzschild radius is normalized to 1.

for the Schwarzschild potential U — and it is this fact which prevents the same analysis in this paper being carried through for (2).

From such considerations it appears that the use of the wave equation with Pöschl-Teller potential is a good starting point for a mathematical investigation of the completeness of quasinormal modes. One may hope that, given the different decay as $x \rightarrow \infty$ the results have some similarities with those for U . This is illustrated in Figure 2, where the solutions of (2) and (4) are compared. In both cases, the initial data describes a gaussian pulse which is purely incoming from infinity. In the figure, the lines show the resulting outgoing waves, as seen by a distant observer. The solid line corresponds to the Pöschl-Teller potential and the dotted line corresponds to the Schwarzschild potential. At early times, the solutions are very similar, although their behaviours differ at late times.

The most difficult and time-consuming part of the calculations for the results on completeness, was in the derivation of the estimates, (32) and (33), on the analytic continuation of the resolvent of the Sturm-Liouville operator with Pöschl-Teller potential. It was not clear *a priori*, from previous works on quasinormal modes, what form the estimates should take in order to prove or disprove these completeness results. Although the estimates are given here only for the Pöschl-Teller potential, one can hope that their structure is representative for other potentials. If this is the case, the form of the estimates (32) and (33) could provide a basis for further completeness calculations for different potentials.

Section 3, which contains the rigorous basis of this paper, is intended to be partly pedagogical. The results apply to a much more general class than just partial differential operators. Although these results can be found in the mathematical literature, they are not easily accessible, and in this section the relevant results are collected and presented in a manner more convenient for quasinormal mode considerations.

A study of the literature on quasinormal modes shows that some of these results (especially (16) and (19)) are already used. However, the form used is often not valid for the case considered, or the proof of its validness is left open. Formulae (16) and (19) in Section 3 offer a rigorous starting point for such considerations in the future. Further, in some more physically motivated papers dealing with quasinormal mode expansions, mathematical terminology such as “convergence” or “completeness”, is used somewhat freely. That is, the terminology is used but corresponding proofs are not given rigorously, or are substituted by “physical” arguments. While important, physical intuition into whether or not an infinite sum converges is very different from, and cannot substitute, a proof of convergence. Hence, in this paper much importance is placed on mathematical rigor.

Now, for those readers who are not concerned with the details of the various proofs, the main results of the paper are summarised. For this, denote by $q(A)$ the set of quasinormal frequencies of V and for each $\omega \in q(A)$ denote by u_ω the corresponding quasinormal eigenfunction. In addition let f be some complex-valued C^∞ -function with compact support and let ϕ_f be the corresponding solution of (4) with initial values

$$\phi_f(0, x) = 0 \quad \text{and} \quad \frac{\partial \phi_f}{\partial t}(0, x) = f(x), \quad (6)$$

for all real x . Finally denote by $\phi_{g,f}$ the following averaged function obtained from ϕ_f ,

$$\phi_{g,f}(t) := \begin{cases} \int_{-\infty}^{+\infty} g^*(x) \cdot \phi_f(t, x) dx & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}, \quad (7)$$

where g is some complex-valued C^∞ -function with compact support. The main results of this paper are,

1. The quasinormal modes of V are complete, in the sense that there is a family of complex numbers c_ω , $\omega \in q(A)$ (given explicitly in Section 5, see (38)) such that for *for a large enough* t_0 and for every $t \in [t_0, \infty)$

$$\left(c_\omega \cdot \int_{-\infty}^{+\infty} u_\omega(y') f(y') dy' \cdot \int_{-\infty}^{+\infty} g^*(x') u_\omega(x') dx' \cdot e^{i\omega t} \right)_{\omega \in q(A)} \quad (8)$$

*is absolutely summable with sum $\phi_{g,f}(t)$. So the summation of this sequence (using any order of summation) gives the quasinormal mode expansion of $\phi_{g,f}$ for large times.*³ In addition, estimates for the possible size of

³Here a remark concerning the role of the test function g might be in order. This test function is mainly for mathematical convenience. Below there is also given a corresponding result on the sum of the sequence, which one gets from (8) by formally substituting f by $\delta(x' - x)$ and g by $\delta(y' - y)$, respectively, for some $x \in \mathbb{R}$ and $y \in \mathbb{R}$. The corresponding sum is then a Green's function (more precisely the so called “commutator -distribution”) which is associated to (4).

such t_0 are given depending on the supports of f and g . It is shown that t_0 can be chosen to be any real number which is greater than some explicitly given real number $M(g, f)$ (see (31) in Section 4).

2. $\phi_{g,f}$ has an analytic extension $\bar{\phi}_{g,f}$ to the strip $(M(g, f), \infty) \times \mathbb{R}$ and the sequence (8) is *uniformly absolutely summable* on $[t_0, \infty) \times K_0$ with sum $\bar{\phi}_{g,f}$ for each $t_0 \in (M(g, f), \infty)$ and each compact subset K_0 of \mathbb{R} . As a consequence the sequence (8) can be termwise differentiated to any order on that strip and the resulting sequence of derivatives is uniformly summable on $[t_0, \infty) \times K_0$ with a sum equal to the corresponding derivative of $\bar{\phi}_{g,f}$.
3. A result shown in Appendix B indicates that the QNM sum exists only for large enough times. There it is shown that the sequence

$$\left(c_\omega [u_\omega(0)]^2 \right)_{\omega \in q(A)}, \quad (9)$$

which one gets formally³ from (8) by the substitutions $t = 0$, f by $\delta(x)$ and g by $\delta(y)$, *is not absolutely summable. Hence for that case there cannot be associated a sum with (9) which is independent of the order of the summation.*

The plan for the remaining part of this paper is the following: In *Section 3* the wave equation (4) is associated with the linear self-adjoint (Sturm-Liouville) operator A (10). A representation of the solution of the initial value problem is given. This representation is found by applying the members of a special family (parameterized by time) of functions of A (which are bounded linear operators) on the data (see e.g. [3]). Using a result of semigroup theory [6] these functions are represented by integrals over the resolvent of A (see (16) or (B22) in Appendix B). Because of the analyticity properties of the resolvent the method of contour integration can be used in *Section 5*. Using the residue theorem the quasinormal frequencies (and modes) come in since they are (in quantum terminology) common poles of the analytic continuations of a set of transition amplitudes of the resolvent (see e.g. [10], Volume IV, page 55). By explicit estimates on these analytic continuations which are supplied in *Section 4* it is then shown that the resonance modes are complete for a large enough time t_0 . In addition estimates for t_0 are given. These bounds depend on the support of the data. *Section 6* gives a discussion of the results. *Appendix A* supplies mathematical details to the results of Sections 3, 4 and 5. Finally, for readers better acquainted with the “Laplace method” [11] than operator theory, *Appendix B* gives a (not completely rigorous) derivation for the basic representation (16) used in this paper for the solution of the initial value problem for (4).

III. AN INITIAL VALUE FORMALISM FOR THE WAVE EQUATION

In order to give (4) a well-defined meaning one has, of course, to specify the differentiability properties of ϕ . In the following a standard abstract approach for giving such a specification is used.⁴ The purpose of this section is the derivation of the representations (16) and (19) of the solutions of the initial-value problem of (4). *These representations are basic for this paper.* The methods for this derivation come from semigroup theory and spectral theory. For the reader not familiar with these methods, this is rederived in Appendix B using the so called “Laplace method” (e.g. [11]).

Define the Sturm-Liouville operator $A : W^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ by

$$Af := -f'' + Vf, \quad (10)$$

for each $f \in W^2(\mathbb{R})$. Here $L^2(\mathbb{R})$ denotes the Hilbert space of complex-valued square integrable functions on the real line with scalar product $\langle | \rangle$ defined by

$$\langle f | g \rangle := \int_{-\infty}^{+\infty} f^*(x) \cdot g(x) dx, \quad (11)$$

⁴See for example, page 295 in Volume II of [10]. Of course there also other approaches for such a specification. Usually, all approaches turn out to be “equivalent” in that the unique solution of the initial value problem in one approach can be reinterpreted in such a way that it coincides with the corresponding one in another approach. The approach chosen in this paper has the advantage that it leads in a natural way to eigenfunction expansions and/or quasinormal eigenfunction expansions of the solution.

for all $f, g \in L^2(\mathbb{R})$; $W^2(\mathbb{R})$ denotes the dense subspace of $L^2(\mathbb{R})$, consisting of two times distributionally differentiable elements, and the distributional derivative is denoted by a prime. By the Rellich-Kato theorem⁵, it follows that A is a densely defined linear and self-adjoint operator in $L^2(\mathbb{R})$ which results from perturbing the linear self-adjoint operator A_0 defined by

$$A_0 := (W^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}), f \mapsto -f'') , \quad (12)$$

by the bounded linear self-adjoint operator with the function V ⁶ (the so called maximal multiplication operator corresponding to V). Further, the spectrum of A consists of all positive real numbers (including zero). The proof of this, which is not difficult, is not given here. The formulation of (4) used in the following is given by

$$\ddot{\phi}(t) = -A\phi(t) , \quad (13)$$

for each $t \in \mathbb{R}$, where ϕ is required to be a C^2 -map from \mathbb{R} into $L^2(\mathbb{R})$ with values in $W^2(\mathbb{R})$, and a dot denotes time differentiation.⁷ Using only abstract properties of A , namely its selfadjointness and its positiveness, it follows from the proposition on page 295 in Volume II of [10] and Theorem 11.6.1 in [6] (see also Theorem 1 in Appendix B) that for each $f \in W^2(\mathbb{R})$ there is a unique $\phi_f \in C^2(\mathbb{R}, L^2(\mathbb{R}))$ with values in $W^2(\mathbb{R})$, satisfying the initial conditions

$$\phi_f(0) = 0 \quad \text{and} \quad \dot{\phi}_f(0) = f , \quad (14)$$

and that the solution ϕ_f has the following representation. Define

$$\phi_{g,f}(t) := \begin{cases} < g | \phi_f(t) > & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} . \quad (15)$$

The representation of ϕ_f is given by

$$\phi_{g,f}(t) = \frac{1}{\sqrt{2\pi}} e^{\epsilon t} (F^{-1} \mathcal{R}_{g,f}(\cdot - i\epsilon))(t) , \quad (16)$$

for (Lebesgue-) almost all $t \in \mathbb{R}$, where ϵ is an, otherwise arbitrary, strictly positive real number; g is an, otherwise arbitrary, element of $L^2(\mathbb{R})$; F is the unitary linear Fourier transformation on⁸ $L^2(\mathbb{R})$ and $\mathcal{R}_{g,f} : \mathbb{R} \times (-\infty, 0) \rightarrow \mathbb{C}$ is defined by

$$\mathcal{R}_{g,f}(\omega) := < g | R(\omega^2) f > , \quad (17)$$

for each $\omega \in \mathbb{R} \times (-\infty, 0)$. Here $R : \mathbb{C} \setminus [0, \infty) \rightarrow L(L^2(\mathbb{R}), L^2(\mathbb{R}))$ is the so called *resolvent* of A , which associates to each $\lambda \in \mathbb{C} \setminus [0, \infty)$ the inverse of the operator $A - \lambda$. $L(L^2(\mathbb{R}), L^2(\mathbb{R}))$ denotes the linear space of bounded linear operators on $L^2(\mathbb{R})$.

Note that $\mathcal{R}_{g,f}(\cdot - i\epsilon)$ is square integrable, as can easily be concluded from the bound

$$|\mathcal{R}_{g,f}(\omega)| \leq \frac{\|f\|_2 \cdot \|g\|_2}{\max\{2|\omega_2| \cdot |\omega_1|, \omega_2^2 - \omega_1^2\}} , \quad (18)$$

which is valid for each $\omega = \omega_1 + i\omega_2 \in \mathbb{R} \times (-\infty, 0)$. This bound requires⁹ also only the self-adjointness and positivity of A .

Finally, using a well-known property of the Fourier transformation¹⁰, it follows from (16) that there exists a subset N of \mathbb{R} having Lebesgue measure zero such that for each $t \in [0, \infty) \setminus N$

$$\phi_{g,f}(t) = \frac{1}{2\pi} \lim_{\nu \rightarrow \infty} \int_{-\nu}^{\nu} e^{it \cdot (\omega - i\epsilon)} \mathcal{R}_{g,f}(\omega - i\epsilon) d\omega . \quad (19)$$

⁵Theorem X.12 in Volume II of [10].

⁶See for example Proposition 1 in Chapter VIII.3, Volume I of [10].

⁷Hence (4) is viewed, similarly as in the case of the Schroedinger equation (but with a second order time derivative), as an ordinary differential equation for a curve in a Hilbert space.

⁸For the definition see Chapter IX in Volume II of [10].

⁹Spectral Theorem VIII.5(b) in Volume I of [10].

¹⁰See e.g. the representation of the Fourier transformation on page 11 in Volume II of [10].

The representations (16) and (19) have here been given for the special case of the Pöschl-Teller potential. In fact, as hinted at in the above text, (16) is an application of the abstract Theorem 1 given at the end of Appendix B, which is far more general. The representation (B21) given in that theorem is, for instance, also valid for wave equations in arbitrary space dimensions.

IV. ANALYTIC PROPERTIES OF THE RESOLVENT

Formula (19) is the starting point of a contour integration, which is performed in Section 5, and eventually leads to the results on the completeness of the quasinormal modes. The basis for that contour integration is provided by the estimates (32), (33) of this section below on the analytic continuation of $\mathcal{R}_{g,f}$. The purpose of this section is mainly to explain these estimates. A sketch of the proofs of these estimates is given in Appendix A.

Let f and g be arbitrary, considered as given from now on, complex-valued C^2 -functions on \mathbb{R} with compact supports. Then it follows from general analytic properties of resolvents that the function $\mathcal{R}_{g,f}$ defined in Equation (17) is an analytic function on the open lower half-plane.

Now *using for the first time*¹¹ *the special properties of the Pöschl-Teller potential* it will be concluded that $\mathcal{R}_{g,f}$ has an analytic extension into the closed upper half-plane. In order to see this the auxiliary function $\bar{\mathcal{R}}_{g,f}$ is now defined,

Define the set $q(A)$ of “quasinormal frequencies of A ” by

$$q(A) := \bigcup_{k \in \mathbb{N}} \{\omega_k^-, \omega_k^+\}, \quad (20)$$

where for each $k \in \mathbb{N}$,

$$\omega_k^- := i \cdot \left(\frac{1}{2} - \alpha + k\right)/b, \quad \omega_k^+ := i \cdot \left(\frac{1}{2} + \alpha + k\right)/b, \quad (21)$$

and

$$\alpha := \begin{cases} \sqrt{\frac{1}{4} - b^2 V_0} & \text{for } b^2 V_0 \leq \frac{1}{4} \\ i\sqrt{b^2 V_0 - \frac{1}{4}} & \text{for } b^2 V_0 > \frac{1}{4} \end{cases}. \quad (22)$$

For each $\omega \in \mathbb{C} \setminus q(A)$ the corresponding $\bar{\mathcal{R}}_{g,f}(\omega)$ is defined by

$$\bar{\mathcal{R}}_{g,f}(\omega) = \iint_{\mathbb{R}^2} g^*(x) K(\omega, x, y) f(y) dx dy, \quad (23)$$

where for each $x, y \in \mathbb{R}$:

$$K(\omega, x, y) = -\frac{1}{W(\omega)} \begin{cases} u_r(\omega, x) u_l(\omega, y) & \text{for } y \leq x \\ u_l(\omega, x) u_r(\omega, y) & \text{for } y > x \end{cases}, \quad (24)$$

and for each $\omega \in \mathbb{C}, x \in \mathbb{R}$:

$$\begin{aligned} u_l(\omega, x) &:= e^{i\omega x} \cdot \bar{F} \left(\frac{1}{2} - \alpha, \frac{1}{2} + \alpha, 1 + ib\omega, \frac{1}{1 + e^{-\frac{2x}{b}}} \right), \\ u_r(\omega, x) &:= u_l(\omega, -x), \end{aligned} \quad (25)$$

and

$$\begin{aligned} W(\omega) &:= u_l(\omega, x) (u_r(\omega, \cdot))'(x) - u_r(\omega, x) (u_l(\omega, \cdot))' \\ &= -\frac{2}{b} \cdot \frac{1}{\Gamma} \left(\frac{1}{2} + \alpha + ib\omega \right) \frac{1}{\Gamma} \left(\frac{1}{2} - \alpha + ib\omega \right). \end{aligned} \quad (26)$$

¹¹Apart from its positivity, which has already been used in concluding that A is a positive operator.

Here $\bar{F} : \mathbb{C}^3 \times U_1(0) \rightarrow \mathbb{C}$ is the analytic extension of the function

$$(\mathbb{C}^2 \times (\mathbb{C} \setminus -\mathbb{N}) \times U_1(0) \rightarrow \mathbb{C}, (a, b, c, z) \mapsto F(a, b, c, z)/\Gamma(c)) , \quad (27)$$

where the hypergeometric function (Gauss series) F and the Gamma function Γ are defined according to [1] and $1/\Gamma$ denotes the extension of $(\mathbb{C} \setminus -\mathbb{N} \rightarrow \mathbb{C}, c \mapsto 1/\Gamma(c))$ to an entire analytic function.

Note that for each $\omega \in \mathbb{C}$ the corresponding functions $u_l(\omega, \cdot)$, $u_r(\omega, \cdot)$ satisfy,

$$\begin{aligned} (u_l(\omega, \cdot))''(x) - (V(x) - \omega^2) \cdot u_l(\omega, x) &= 0 , \\ (u_r(\omega, \cdot))''(x) - (V(x) - \omega^2) \cdot u_r(\omega, x) &= 0 , \end{aligned} \quad (28)$$

for each $x \in \mathbb{R}$. In addition for each $\omega \in \mathbb{R} \times (-\infty, 0)$ the associated $u_l(\omega, \cdot)$, $u_r(\omega, \cdot)$ is \mathcal{L}^2 near $-\infty$ and $+\infty$, respectively. Using this, along with general results on “Sturm-Liouville” operators (see e.g. [13]) and differentiation under the integral sign, it follows that $\bar{\mathcal{R}}_{g,f}$ is an analytic function on $\mathbb{C} \setminus q(A)$, which coincides with $\mathcal{R}_{g,f}$ on the open lower half-plane. The proof of this is elementary and not given in this paper.

The QNF of A , which coincide with the zeros of the Wronskian determinant function W , are poles of $\bar{\mathcal{R}}_{g,f}$. These poles are simple for the case $\alpha \neq 0$ and second order for the case $\alpha = 0$. In somewhat misleading, but common mathematical terminology, such poles are often called “second sheet poles of the resolvent (of A)” or “resonances” (of A) (see for example Volume IV of [10]). This terminology is somewhat misleading, because they not only depend on A , but also on the choice of a dense subspace of \mathcal{L}^2 (see for example Volume IV of [10]), which is here the space of complex-valued C^∞ - functions on the real line with compact support, which is the space from where the data for (4) are taken.

The QNM corresponding to the QNF of A , $u_r(\omega, \cdot)u_l(\omega, \cdot)$, $\omega \in q(A)$ satisfy,

$$\begin{aligned} u_r(\omega_k^\pm, x) &= (-1)^k \cdot u_l(\omega_k^\pm, x) \\ u_{\omega_k^\pm}(x) &:= u_l(\omega_k^\pm, x) = \frac{1}{\Gamma(\frac{1}{2} \mp \alpha - k)} \cdot (2 \cosh(x/b))^{\frac{1}{2} \pm \alpha + k} \cdot \\ &\quad F\left(-k, -k \mp 2\alpha, \frac{1}{2} \mp \alpha - k, 1/(1 + e^{-2x/b})\right) , \end{aligned} \quad (29)$$

for each $k \in \mathbb{N}$, $x \in \mathbb{R}$. This result is also easy to see and its proof is not given in this paper.

In view of the analytic properties of $\mathcal{R}_{g,f}$ it is natural to try to evaluate the right hand side of (19) by contour integration. This is done in the next section and that contour integration leads to the completeness results of this paper on the QNM of the Pöschl-Teller potential. The basis for the contour integration is provided by estimates on $\bar{\mathcal{R}}_{g,f}$ which are now given.

The estimates depend on the parameters $d(g, f)$, $m(g, f)$ and $M(g, f)$, which define certain “distances” between the supports of g and f . These distances are defined by,

$$\begin{aligned} d(g, f) &:= \min\{|x - y| : x \in \text{supp}(g) \text{ and } y \in \text{supp}(f)\} , \\ m(g, f) &:= \max\{|x - y| : x \in \text{supp}(g) \text{ and } y \in \text{supp}(f)\} , \\ M(g, f) &:= \max\{D(x, y) : x \in \text{supp}(g) \text{ and } y \in \text{supp}(f)\} \geq m(g, f) , \end{aligned} \quad (30)$$

where

$$D(x, y) := |x - y| + b \cdot \begin{cases} \ln(1 + 2e^{-2x/b}) + \ln(1 + 2e^{2y/b}) & \text{for } y \leq x \\ \ln(1 + 2e^{-2y/b}) + \ln(1 + 2e^{2x/b}) & \text{for } y > x \end{cases} , \quad (31)$$

for each $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Note that the quantities $d(g, f)$ and $m(g, f)$ have an obvious geometrical interpretation.

The following estimates hold for $\bar{\mathcal{R}}_{g,f}$ and each $\omega = \omega_1 + i\omega_2 \in \mathbb{C} \setminus (q(A) \cup -q(A))$

$$|\bar{\mathcal{R}}_{g,f}(\omega)| \leq$$

$$C_1(g, f) \cdot \frac{e^{2\pi b|\omega_1|}}{|\cos(2\pi\alpha) + \cosh(2\pi b\omega)|} \cdot (1 + 4b^2\omega_1^2)^{-1/2} \cdot \begin{cases} e^{\omega_2 \cdot d(g, f)} & \text{for } \omega_2 < 0 \\ e^{\omega_2 \cdot M(g, f)} & \text{for } \omega_2 \geq 0 \end{cases} , \quad (32)$$

and if in addition both $\text{supp}(f) \subset [0, \infty)$ and $\text{supp}(g) \subset (-\infty, 0]$ or $\text{supp}(f) \subset (-\infty, 0]$ and $\text{supp}(g) \subset [0, \infty)$:

$$|\bar{\mathcal{R}}_{g,f}(\omega)| \leq$$

$$C_2(g, f) \cdot \frac{e^{2\pi b|\omega_1|}}{|\cos(2\pi\alpha) + \cosh(2\pi b\omega)|} \cdot (1 + 4b^2\omega_1^2)^{-1/2} \cdot \begin{cases} e^{\omega_2 \cdot d(g, f)} & \text{for } \omega_2 < 0 \\ e^{\omega_2 \cdot m(g, f)} & \text{for } \omega_2 \geq 0 \end{cases} , \quad (33)$$

where $C_1(g, f), C_2(g, f) \in [0, \infty)$ are given in Appendix A.

The derivation of (32) and (33) is given in Appendix A. They were obtained by different methods of estimation. Note that depending on the methods used in their derivation these estimates are “singular” in the open lower half-plane at the elements of $-q(A)$, although $\bar{\mathcal{R}}_{g,f}$ is analytic there. This will not be relevant in the following. From (32) and (33) follows, in particular, that the *restriction of $\bar{\mathcal{R}}_{g,f}$ to the real axis is square integrable*. This is used in the contour integration in the next section. Note that the *corresponding statement is false* for the operator A_0 (see (12) for the definition) although it is only a bounded (i.e. in the operator theoretic sense “very small”) perturbation of A . For this case the corresponding $\bar{\mathcal{R}}_{g,f}$ is analytic on $\mathbb{C} \setminus \{0\}$ and has in general a first order pole at $\omega = 0$.

V. CONSEQUENCES

A first implication of the estimates (32) and (33) is, roughly speaking that, for the special case of the Pöschl-Teller potential, the formula (16) is also true for the case $\epsilon = 0$, making subsequent contour integration easier.

This can be seen as follows. The estimates (32) and (33) imply the boundedness of the function which associates the value $\|\mathcal{R}_{g,f}(\cdot + i\omega_2)\|_2$ to each $\omega_2 \in (-\infty, 0)$, where $\|\cdot\|_2$ denotes the norm which is induced on $L^2(\mathbb{R})$ by the scalar product $\langle \cdot | \cdot \rangle$. Hence it follows by a Paley-Wiener theorem¹² that the sequence $(\mathcal{R}_{g,f}(\cdot + i\omega_2))_{\omega_2 \in (-\infty, 0)}$ converges for $\omega_2 \rightarrow 0$ in $L^2(\mathbb{R})$ to the restriction $\mathcal{R}_{g,f}|_{\mathbb{R}}$ of $\mathcal{R}_{g,f}$ to the real axis.

Hence (16) and the continuity of the Fourier transformation leads to

$$\phi_{g,f} = \frac{1}{\sqrt{2\pi}} \cdot F^{-1} \bar{\mathcal{R}}_{g,f}|_{\mathbb{R}}. \quad (34)$$

Using a well-known result in the theory of the Fourier transformation¹⁰ it follows that there exists a subset N of \mathbb{R} having Lebesgue measure zero such that for each $t \in [0, \infty) \setminus N$

$$\phi_{g,f}(t) = \frac{1}{2\pi} \cdot \lim_{\nu \rightarrow \infty} \int_{-\nu}^{\nu} e^{it\omega} \cdot \bar{\mathcal{R}}_{g,f}(\omega) d\omega. \quad (35)$$

In particular this implies that $\phi_{g,f}$ is square integrable — the corresponding statement not being generally true when the operator A is replaced by A_0 .

Equation (35) can now be contour integrated, using the Cauchy integral theorem and Cauchy integral formula, to give an expansion of $\phi_{g,f}$ with respect to the QNM. *In the following, for convenience, the case $\alpha = 0$ is excluded.* Then the QNF of A , are simple poles of $\bar{\mathcal{R}}_{g,f}$. But with the help of (32) and (33) the same contour integration can also be carried through for the case $\alpha = 0$ leading to similar results.

The contours are chosen as the boundaries of the rectangles with corners

$$\begin{aligned} &(-\nu, 0), (\nu, 0), (\nu, n/b), (-\nu, n/b) \quad \text{and} \\ &(-\nu, 0), (\nu, 0), (\nu, -n/b), (-\nu, -n/b) \end{aligned}$$

where ν is an integer and n is a natural number. Then, following from (32) and (33), the integrals along the paths in the upper and lower half plane vanish for certain t in the limit when first $\nu \rightarrow \infty$ and then $n \rightarrow \infty$. The calculations for this are elementary, but lengthy, and will not be carried through in this paper, only their results will be given in the following.

In particular, as demanded by causality, the function $\phi_{g,f}$ vanishes on the interval $[0, d_{g,f}]$, as is seen by closing the contour in the lower half plane.

Closing the contour in the upper half plane leads to two statements concerning the expansion of $\phi_{g,f}$ in the QNM. First define μ by

$$\mu := \begin{cases} m(g, f) & \text{if } \begin{cases} \text{either } \text{supp}(f) \subset [0, \infty) \text{ and } \text{supp}(g) \subset (-\infty, 0] \\ \text{or } \text{supp}(f) \subset (-\infty, 0] \text{ and } \text{supp}(g) \subset [0, \infty) \end{cases} \\ M(g, f) & \text{otherwise} \end{cases}. \quad (36)$$

¹²see for example Theorems 1 and 2 in Section 4, Chapter VI of [14].

Now define for each $n \in \mathbb{N}$ and $t \in \mathbb{C}$ the entire analytic function $s_{g,f,n}$ by

$$s_{g,f,n}(t) := \sum_{k=0}^n \left(c_{\omega_k^-} \int_{-\infty}^{+\infty} u_{\omega_k^-}(y) f(y) dy \int_{-\infty}^{+\infty} g^*(x) u_{\omega_k^-}(x) dx e^{i\omega_k^- t} \right. \\ \left. + c_{\omega_k^+} \int_{-\infty}^{+\infty} u_{\omega_k^+}(y) f(y) dy \int_{-\infty}^{+\infty} g^*(x) u_{\omega_k^+}(x) dx e^{i\omega_k^+ t} \right), \quad (37)$$

where for each $k \in \mathbb{N}$

$$c_{\omega_k^-} := \frac{(-1)^k \pi / (2 \sin(2\pi\alpha))}{\Gamma(1+k) \Gamma(1-2\alpha+k)}, \\ c_{\omega_k^+} := \frac{(-1)^{k+1} \pi / (2 \sin(2\pi\alpha))}{\Gamma(1+k) \Gamma(1+2\alpha+k)}. \quad (38)$$

The following statements (i) and (ii) are then true.

- (i) For each $t_0 \in (\mu, \infty)$ the sequence $(s_{g,f,n})_{n \in \mathbb{N}}$ converges on $[t_0, \infty)$ in the \mathcal{L}^2 -mean to $\phi_{g,f}$.
- (ii) The restriction of $\phi_{g,f}$ to (μ, ∞) has an extension to an analytic function on the strip $(\mu, \infty) \times \mathbb{R}$. For each $t_0 \in (\mu, \infty)$ and each compact subset K_0 of \mathbb{R} the sequence $(s_{g,f,n})_{n \in \mathbb{N}}$ converges uniformly on $[t_0, \infty) \times K_0$ to this extension.

Note that in these results a special order of the summation for the QNM sequence (37) is used, which is induced by the chosen contour in the integration. That this result is independent of this order of summation for $\mu := M(g, f)$ follows from further results on the summability of the sequence

$$(c_\omega u_\omega(y) u_\omega(x) e^{i\omega t})_{\omega \in q(A)}, \quad (39)$$

for given $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $t \in [0, \infty)$, which are now stated. The corresponding proofs are given in Appendix B. There, it is shown by direct estimates on the sequence elements that, given $x \in \mathbb{R}$ and $y \in \mathbb{R}$, this sequence is *absolutely and uniformly summable* on $[t_0, \infty) \times K_0$ where $t_0 > D_s(x, y)$, K_0 is any compact subset of \mathbb{R} and

$$D_s(x, y) := b \log \left(2 \left[\cosh \left(\frac{x-y}{b} \right) + \cosh \left(\frac{x+y}{b} \right) \right] \right). \quad (40)$$

Hence, in particular follows the analyticity of the function which associates to each $t \in (D_s(x, y), \infty) \times \mathbb{R}$ the value

$$\sum_{\omega \in q(A)} c_\omega u_\omega(y) u_\omega(x) e^{i\omega t}. \quad (41)$$

Further it is shown in Appendix B that

$$\left(c_\omega \cdot \int_{-\infty}^{+\infty} u_\omega(y) f(y) dy \cdot \int_{-\infty}^{+\infty} g^*(x) u_\omega(x) dx \cdot e^{i\omega t} \right)_{\omega \in q(A)}, \quad (42)$$

is *absolutely and uniformly summable* on $[t_0, \infty) \times K_0$, where $t_0 > M_s(g, f)$, K_0 is any compact subset of \mathbb{R} and

$$M_s(g, f) := \max\{D_s(x, y) : x \in \text{supp}(g) \text{ and } y \in \text{supp}(f)\}. \quad (43)$$

It is easily seen that

$$D(x, y) \geq D_s(x, y), \quad (44)$$

for all $x \in \mathbb{R}$ and $y \in \mathbb{R}$ and hence that

$$M(g, f) \geq M_s(g, f). \quad (45)$$

Using the results on the QNM sequence from this section above, it follows that for every $t_0 > M(g, f)$ and for every $t \in [t_0, \infty)$ the sequence (42) is *absolutely summable with sum* $\phi_{g,f}(t)$ and that the sequence (42) is *uniformly absolutely summable* on $[t_0, \infty) \times K_0$ with sum $\phi_{g,f}$ for each $t_0 \in (M(g, f), \infty)$ and each compact subset K_0 of \mathbb{R} . As a consequence the sequence (42) can be termwise differentiated to any order on that strip and the resulting sequence of derivatives is uniformly summable on $[t_0, \infty) \times K_0$ with a sum equal to the corresponding derivative of $\phi_{g,f}$.

A further result shown in Appendix B indicates that the QNM sum exists only for large enough times. There it is shown that, for the special case of $x = y = 0$ and $t = 0 (< D_s(0, 0))$ the sequence (39) is *not absolutely summable* because the sum

$$\sum_{k=0}^n \left| c_{\omega_k^-} \left[u_{\omega_k^-}(0) \right]^2 \right|, \quad (46)$$

is shown to *diverge* for $n \rightarrow \infty$. Hence, for that case, there can be no associated sum with (39) which is independent of the order of the summation.

VI. DISCUSSION AND OPEN QUESTIONS

In this paper we gave several results on the completeness of the quasinormal modes of the Pöschl-Teller potential. A main result is that any solution of the wave equation with the Pöschl-Teller potential (4) corresponding to C^∞ -data with compact support can be expanded uniformly in time with respect to the quasinormal modes after a large enough time t_0 . Further the corresponding series are absolutely convergent, and hence do not depend on the order of summation. In addition we showed that these series can be arbitrarily often termwise partially differentiated with respect to time, again leading to series which converge absolutely and uniformly in time on $[t_0, \infty)$ to the corresponding time derivatives of the solution. Estimates of t_0 were given which depend on the support of the data and on the point of observation.

Estimates were also given for the time t_1 from when the solution can be expanded uniformly in time with respect to the quasinormal modes, *where a special order of summation is assumed*. Also for this case the quasinormal mode series can be arbitrarily often termwise partially differentiated with respect to time thereby leading to series which converge uniformly in time on $[t_1, \infty)$ to the corresponding time derivatives of the solution of the initial value problem.

These estimates have in common that they depend on both the support properties of the data and the point of observation, and that they are greater or equal to the geometrical distance between the support of the data and the observational point.

We showed that, for an “early” time and zero distance between the support of the data and observational point, the corresponding quasinormal mode series is *not* absolutely convergent. Hence, there is no associated sum, since in general different orders of summation will give different results.

Several open questions remain. The results of this paper suggest a relationship between the convergence of the quasinormal mode sums of the Pöschl-Teller potential and causality. To make this clearer, one would like to have a complete overview of the convergence of the quasinormal mode sums depending on the support of the data as well as the point of observation; possibly depending on whether a special order of summation is assumed or not and possibly depending on whether the series converges to the corresponding solution of the wave equation or not.

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APPENDIX A:

This appendix gives a derivation of the estimates (32) and (33) as well as an estimate on the members of the sequence (39). All these estimates are crucial for the proof of the expansion formulae in Section 4. The definitions and the notation of [1] are used throughout. The derivation uses the following auxiliary estimate.

Lemma 1. Let $n \in \mathbb{N}$, $a \in (0, 1)$ and $s \in [0, \infty)$ be given. Then

$$\int_0^{\pi/2} e^{-st} \sin^{n+a-1}(t) dt \leq \frac{\pi B_a}{2} \cdot \frac{2^{-n} \Gamma(n+1)}{(\Gamma(\frac{n}{2}+1))^2} \cdot (1+s^2)^{-a/2}, \quad (\text{A1})$$

where

$$B_a := \left(\frac{4}{\pi}\right)^a \cdot \max \left\{ 2^{-a} \cdot \left(\pi + \frac{1}{a}\right), \Gamma(a) + \pi \cdot \left(\frac{a}{e}\right)^a \right\}. \quad (\text{A2})$$

Note that later on, (A1) has to provide a proper estimate for the vanishing of the integral both for $s \rightarrow \infty$ and $n \rightarrow \infty$. This demand excludes, for instance, an application of the method of partial integration, in the following proof.

Proof. First by standard estimates for the sine-function one gets

$$\begin{aligned} & \int_0^{\pi/2} e^{-st} \sin^{n+a-1}(t) dt \\ & \leq \left(\frac{2}{\pi}\right)^{a-1} \cdot \int_0^{\pi/2} e^{-st} t^{a-1} \sin^n(t) dt \\ & \leq \left(\frac{2}{\pi}\right)^{a-1} \cdot \left[2^{-n} \cdot \int_0^{1/2} e^{-st} t^{a-1} dt + 2^{1-a} \cdot e^{-s/2} \cdot \int_{1/2}^{\pi/2} \sin^n(t) dt \right] \\ & \leq \left(\frac{2}{\pi}\right)^{a-1} \cdot \int_0^{\pi} \sin^n(t) dt \cdot \left[\frac{1}{\pi} \cdot \int_0^{1/2} e^{-st} t^{a-1} dt + 2^{-a} \cdot e^{-s/2} \right] \\ & = \left(\frac{2}{\pi}\right)^{a-1} \cdot \frac{2^{-n} \Gamma(n+1)}{(\Gamma(\frac{n}{2}+1))^2} \cdot \left[\int_0^{1/2} e^{-st} t^{a-1} dt + \pi \cdot 2^{-a} \cdot e^{-s/2} \right], \end{aligned} \quad (\text{A3})$$

where in the last equality the identity

$$\int_0^{\pi} \sin^n(t) dt = \frac{\pi \cdot \Gamma(n+1)}{2^n \cdot (\Gamma(\frac{n}{2}+1))^2}, \quad (\text{A4})$$

was used¹³. For the case $0 \leq s \leq 1$ one has now,

$$\int_0^{1/2} e^{-st} t^{a-1} dt + \pi \cdot 2^{-a} \cdot e^{-s/2} \leq \int_0^{1/2} t^{a-1} dt + \pi \cdot 2^{-a} \leq \left(\pi + \frac{1}{a}\right) \cdot (1+s^2)^{-a/2}, \quad (\text{A5})$$

and for the case $s > 1$,

$$\begin{aligned} & \int_0^{1/2} e^{-st} t^{a-1} dt + \pi \cdot 2^{-a} \cdot e^{-s/2} \leq s^{-a} \cdot [\Gamma(a) + \pi \cdot 2^{-a} \cdot s^a e^{-s/2}] \\ & \leq s^{-a} \cdot \left[\Gamma(a) + \pi \cdot \left(\frac{a}{e}\right)^a \right] \\ & \leq 2^a \cdot \left[\Gamma(a) + \pi \cdot \left(\frac{a}{e}\right)^a \right] \cdot (1+s^2)^{-a/2}. \end{aligned} \quad (\text{A6})$$

The result (A1) then follows from (A3), (A5) and (A6). ■

The starting point for the derivation of the formulae (32) and (33) is the following.

Lemma 2. Let $\omega \in \mathbb{C} \setminus (q(A) \cup -q(A))$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$ be given then

$$K(\omega, x, y) = \frac{\pi^2 b e^{-i\omega \cdot |x-y|}}{\cos(2\pi\alpha) + \cosh(2\pi b\omega)} \cdot \begin{cases} h(\omega, \alpha, (1+e^{2x/b})^{-1}) \cdot h(\omega, -\alpha, (1+e^{-2y/b})^{-1}) & \text{for } y \leq x \\ h(\omega, -\alpha, (1+e^{-2x/b})^{-1}) \cdot h(\omega, \alpha, (1+e^{2y/b})^{-1}) & \text{for } y > x \end{cases}, \quad (\text{A7})$$

¹³This can be derived, for instance, using Formulae 6.2.1, 6.2.2, 6.1.8 and 6.1.18 of [1].

where for arbitrary $\beta \in (-1/2, 1/2) \times \mathbb{R}$ and $x' \in \mathbb{R}$,

$$h(\omega, \beta, x') := \frac{\bar{F}(\frac{1}{2} - \beta, \frac{1}{2} + \beta, 1 + ib\omega, x')}{\Gamma(\frac{1}{2} + \beta - ib\omega)}, \quad (\text{A8})$$

Proof. The proof consists of a straightforward calculation starting from (24) and using (26) in addition to Equations 6.1.17 and 15.1.1 of [1]. ■

Note that the main reason for representing $K(\cdot, x, y)$ in the form (A7) is that only the first elementary factor is singular at the elements of $q(A)$. The price for this is that this factor is singular also at the points of $-q(A)$ in the open upper half-plane. But this will play no role in the following.

The function h satisfies the following estimate, which eventually leads to (A6).

Lemma 3. Let $\beta \in \mathbb{C}$ with $-1/2 < \text{Re}(\beta) < 1/2$, $\omega = \omega_1 + i\omega_2 \in \mathbb{C}$ such that $\omega \neq in/b$ for all $n \in \mathbb{N} \setminus \{0\}$ as well as $\omega \neq -i \cdot (n + \frac{1}{2} + \beta)/b$ for all $n \in \mathbb{N} \setminus \{0\}$ and $x \in (0, \frac{1}{2})$ be given. Then,

$$|h(\omega, \beta, x)| \leq C_\beta \cdot (1 + 4b^2\omega_1^2)^{-\frac{1}{2} \cdot (\frac{1}{2} + \text{Re}(\beta))} \cdot e^{\pi b|\omega_1|} \cdot (1 - 2x)^{-(\frac{1}{2} - \text{Re}(\beta))}, \quad (\text{A9})$$

where

$$C_\beta := \pi^{-1} \cdot 2^{-\frac{1}{2} + \text{Re}(\beta)} \cdot \Gamma\left(\frac{1}{2} - \text{Re}(\beta)\right) \cdot |\cos(\pi\beta)| \cdot e^{\pi \cdot |\text{Im}(\beta)|/2} \cdot B_{\frac{1}{2} + \text{Re}(\beta)}. \quad (\text{A10})$$

Proof. First, using the power series expansion of the hypergeometric function and Equation 6.1.22 of [1], one gets

$$\begin{aligned} |h(\omega, \beta, x)| &= \left| \frac{1}{\Gamma(\frac{1}{2} + \beta - ib\omega) \cdot \Gamma(1 + ib\omega)} \cdot \sum_{n=0}^{\infty} \frac{(\frac{1}{2} - \beta)_n \cdot (\frac{1}{2} + \beta)_n}{(1 + ib\omega)_n} \cdot \frac{x^n}{\Gamma(n+1)} \right| \\ &\leq \frac{1}{|\Gamma(\frac{1}{2} + \beta)|} \cdot \sum_{n=0}^{\infty} \left| \frac{\Gamma(n + \frac{1}{2} + \beta)}{\Gamma(n+1 + ib\omega) \cdot \Gamma(\frac{1}{2} + \beta - ib\omega)} \right| \cdot \left| \left(\frac{1}{2} - \beta\right)_n \cdot \frac{x^n}{\Gamma(n+1)} \right|. \end{aligned} \quad (\text{A11})$$

Using the Formula¹⁴

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} e^{iyt} \cdot \cos^{u-1}(t) dt &= e^{i\pi y/2} \cdot \int_0^\pi e^{-iyt} \cdot \sin^{u-1}(t) dt \\ &= \frac{\pi \cdot 2^{1-u} \cdot \Gamma(u)}{\Gamma(\frac{1+u-y}{2}) \cdot \Gamma(\frac{1+u+y}{2})}, \end{aligned} \quad (\text{A12})$$

which is valid for arbitrary $u \in (0, \infty) \times \mathbb{R}$ and $y \in \mathbb{C}$ (where the expression which includes the Gamma functions is defined by analytic continuation for the cases $y = \pm(1+u)$), one gets in a second step,

$$\begin{aligned} &\left| \frac{\Gamma(n + \frac{1}{2} + \beta)}{\Gamma(n+1 + ib\omega) \cdot \Gamma(\frac{1}{2} + \beta - ib\omega)} \right| \\ &\leq \frac{1}{\pi} \cdot 2^{n - \frac{1}{2} + \text{Re}(\beta)} \cdot \int_{-\pi/2}^{\pi/2} e^{(2b\omega_1 - \text{Im}(\beta)) \cdot t} \cdot \cos^{n - \frac{1}{2} + \text{Re}(\beta)}(t) dt \\ &\leq \frac{1}{\pi} \cdot 2^{n + \frac{1}{2} + \text{Re}(\beta)} \cdot e^{\pi \cdot |\text{Im}(\beta)|/2} \cdot \int_{-\pi/2}^0 e^{-2b|\omega_1| \cdot t} \cdot \cos^{n - \frac{1}{2} + \text{Re}(\beta)}(t) dt \\ &= \frac{1}{\pi} \cdot 2^{n + \frac{1}{2} + \text{Re}(\beta)} \cdot e^{\pi \cdot |\text{Im}(\beta)|/2} \cdot e^{\pi b|\omega_1|} \cdot \int_0^{\pi/2} e^{-2b|\omega_1| \cdot t} \cdot \sin^{n - \frac{1}{2} + \text{Re}(\beta)}(t) dt \\ &\leq 2^{n - \frac{1}{2} + \text{Re}(\beta)} \cdot e^{\pi \cdot |\text{Im}(\beta)|/2} \cdot B_{\frac{1}{2} + \text{Re}(\beta)} \cdot (1 + 4b^2\omega_1^2)^{-\frac{1}{2} \cdot (\frac{1}{2} + \text{Re}(\beta))} \cdot e^{\pi b|\omega_1|}. \end{aligned} \quad (\text{A13})$$

¹⁴See e.g. Equation 5.25 in [7].

With help from Equations 6.1.22 and 6.1.26 of [1] one has for an arbitrary $n \in \mathbb{N}$,

$$\begin{aligned} \left| \left(\frac{1}{2} - \beta \right)_n \right| &= \left| \frac{\Gamma(n + \frac{1}{2} - \beta)}{\Gamma(\frac{1}{2} - \beta)} \right| \leq \frac{\Gamma(n + \frac{1}{2} - \operatorname{Re}(\beta))}{|\Gamma(\frac{1}{2} - \beta)|} \\ &= \frac{\Gamma(\frac{1}{2} - \operatorname{Re}(\beta))}{|\Gamma(\frac{1}{2} - \beta)|} \cdot \left(\frac{1}{2} - \operatorname{Re}(\beta) \right)_n. \end{aligned} \quad (\text{A14})$$

Finally, (A9) follows from (A11), (A13), (A14) with the help of Formulae 6.1.17 and 3.6.8 of [1]. ■

From (A7), (A9) and the continuity of K one gets now the following estimate.

Lemma 4. Let $\omega \in \mathbb{C} \setminus (q(A) \cup -q(A))$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$ be given. Then,

$$\begin{aligned} |K(\omega, x, y)| &\leq \pi^2 b C_\alpha C_{-\alpha} \cdot \frac{e^{2\pi b |\omega_1|}}{|\cos(2\pi\alpha) + \cosh(2\pi b \omega)|} \cdot (1 + 4b^2 \omega_1^2)^{-\frac{1}{2}} \cdot e^{\omega_2 \cdot |x-y|} \cdot \\ &\quad \begin{cases} (\tanh(x/b))^{-(\frac{1}{2}-\operatorname{Re}(\alpha))} \cdot (\tanh(-y/b))^{-(\frac{1}{2}+\operatorname{Re}(\alpha))} & \text{if } x > 0 \text{ and } y < 0 \\ (\tanh(-x/b))^{-(\frac{1}{2}+\operatorname{Re}(\alpha))} \cdot (\tanh(y/b))^{-(\frac{1}{2}-\operatorname{Re}(\alpha))} & \text{if } x < 0 \text{ and } y > 0 \end{cases}. \end{aligned} \quad (\text{A15})$$

From this one gets easily (33) (compare (33) and in particular the assumptions on f and g), where

$$C_2(g, f) := \pi^2 b C_\alpha C_{-\alpha} \cdot \iint_{\mathbb{R}^2} g^*(x) H_2(x, y) f(y) dx dy, \quad (\text{A16})$$

and where

$$H_2(x, y) := \begin{cases} (\tanh(x/b))^{-(\frac{1}{2}-\operatorname{Re}(\alpha))} \cdot (\tanh(-y/b))^{-(\frac{1}{2}+\operatorname{Re}(\alpha))} & \text{if } x > 0 \text{ and } y < 0 \\ (\tanh(-x/b))^{-(\frac{1}{2}+\operatorname{Re}(\alpha))} \cdot (\tanh(y/b))^{-(\frac{1}{2}-\operatorname{Re}(\alpha))} & \text{if } x < 0 \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases}. \quad (\text{A17})$$

A further estimate of the function h uses an integral representation of the hypergeometric function F , which could not be found in the tables on special functions. For this reason that representation and its proof is given now.

Lemma 5. Let $a \in \mathbb{C}$, $b \in \mathbb{C}$, $c \in \mathbb{C} \setminus (-\mathbb{N})$ and $z \in \mathbb{C}$ with $|z| < 1$ be given. Then

(i) if in addition $\operatorname{Re}(c) > \operatorname{Re}(b)$ and $b \notin \mathbb{N} \setminus \{0\}$ hold,

$$\begin{aligned} F(a, b, c, z) &= \pi^{-1} \cdot 2^{c-b-1} \cdot e^{i\pi(c+b-1)/2} \cdot \frac{\Gamma(c) \cdot \Gamma(1-b)}{\Gamma(c-b)} \cdot \\ &\quad \int_0^\pi e^{-i \cdot (c+b-1) \cdot t} \cdot \sin^{c-b-1}(t) \cdot (1 - ze^{-2it})^{-a} dt. \end{aligned} \quad (\text{A18})$$

(ii) if in addition $\operatorname{Re}(b) > 0$ and $c - b \notin \mathbb{N} \setminus \{0\}$ hold,

$$\begin{aligned} F(a, b, c, z) &= \pi^{-1} \cdot 2^{b-1} \cdot e^{i\pi(2c-b-1)/2} \cdot \frac{\Gamma(c) \cdot \Gamma(b-c+1)}{\Gamma(b)} \cdot (1-z)^{c-(a+b)} \cdot \\ &\quad \int_0^\pi e^{-i \cdot (2c-b-1) \cdot t} \cdot \sin^{b-1}(t) \cdot (1 - ze^{-2it})^{a-c} dt. \end{aligned} \quad (\text{A19})$$

Proof. Part (ii) is a direct consequence of (i) and Formula 15.3.3 in [1]. Hence it remains to prove part (i). For this let $\operatorname{Re}(c) > \operatorname{Re}(b)$ and $b \notin \mathbb{N} \setminus \{0\}$. First by Formulae 6.1.22 and 6.1.17 in [1] as well as by some elementary reasoning it follows that

$$\frac{(b)_n}{(c)_n} = (-1)^n \cdot \frac{\Gamma(c) \cdot \Gamma(1-b)}{\Gamma(c-b)} \cdot \frac{\Gamma(c-b)}{\Gamma(c+n) \cdot \Gamma(1-(b+n))}, \quad (\text{A20})$$

where the right hand side is defined by analytic continuation (and hence by zero) for the cases where $b \in \{-n + 1, -n + 2, \dots\}$. From the definition of F (Formula 15.1.1 in [1]), and using (A12) and (A20) follows,

$$\begin{aligned} F(a, b, c, z) &= \frac{\Gamma(c) \cdot \Gamma(1-b)}{\Gamma(c-b)} \cdot \sum_{n=0}^{\infty} \frac{\Gamma(c-b)}{\Gamma(c+n) \cdot \Gamma(1-(b+n))} \cdot (a)_n \cdot \frac{(-z)^n}{\Gamma(n+1)} \\ &= \pi^{-1} \cdot 2^{c-b-1} \cdot e^{i\pi(c+b-1)/2} \cdot \frac{\Gamma(c) \cdot \Gamma(1-b)}{\Gamma(c-b)} \cdot \\ &\quad \lim_{N \rightarrow \infty} \int_0^{\pi} e^{-i \cdot (c+b-1) \cdot t} \cdot \sin^{c-b-1}(t) \cdot \left(\sum_{n=0}^N \frac{(a)_n}{\Gamma(n+1)} \cdot (e^{-2it} \cdot z)^n \right) dt. \end{aligned} \quad (\text{A21})$$

From this (A18) follows using Lebesgue's dominated convergence theorem and the complex version of Formula 3.6.9 ("binomial series") of [1]. ■

Note that part (i) of the foregoing Lemma 5 gives an integral representation for the hypergeometric series for a larger class of parameter values than Formula 15.3.1 in [1], since it does not assume that $\text{Re}(b) > 0$ holds. This will be essential for the derivation of (32).

Actually used in the following is the subsequent corollary of Lemma 5,

Corollary 6. Let $a \in \mathbb{C}$, $b \in (0, \infty) \times \mathbb{R}$, $c \in \mathbb{C} \setminus (-\mathbb{N})$ such that $c - b \notin \mathbb{N} \setminus \{0\}$ and $x \in (-1, 1)$ be given. Then

$$\begin{aligned} F(a, b, c, x) &= \pi^{-1} \cdot 2^{b-1} \cdot \frac{\Gamma(c) \cdot \Gamma(b-c+1)}{\Gamma(b)} \cdot (1-x)^{c-(a+b)} \cdot \\ &\quad \int_{-\pi/2}^{\pi/2} e^{-i \cdot (2c-b-1) \cdot t} \cdot \cos^{b-1}(t) \cdot (x + e^{2it})^{a-c} dt. \end{aligned} \quad (\text{A22})$$

Proof. The relation (A22) follows from (A19) by a straightforward substitution and from the identity

$$(1 + xe^{-2it})^{a-c} = e^{-2i \cdot (a-c) \cdot t} \cdot (x + e^{2it})^{a-c}, \quad (\text{A23})$$

for each $t \in (-\pi/2, \pi/2)$. The latter can easily be shown by analytic continuation. ■

Now with the help of these auxiliary results a further estimate for the function h will be proved, which eventually leads to (32).

Lemma 7. Let $\beta \in \mathbb{C}$ with $-1/2 < \text{Re}(\beta) < 1/2$, $\omega = \omega_1 + i\omega_2 \in \mathbb{C}$ such that $\omega \neq in/b$ for all $n \in \mathbb{N} \setminus \{0\}$ as well as $\omega \neq -i \cdot (n + \frac{1}{2} + \beta)/b$ for all $n \in \mathbb{N} \setminus \{0\}$ and $x \in (0, 1)$ be given. Then,

$$\begin{aligned} |h(\omega, \beta, x)| &\leq C'_\beta \cdot (1 + 4b^2\omega_1^2)^{-\frac{1}{2} \cdot (\frac{1}{2} + \text{Re}(\beta))} \cdot e^{\pi b |\omega_1|} \cdot (1-x)^{-(\frac{1}{2} + \text{Re}(\beta))} \cdot \\ &\quad \begin{cases} \left(\frac{1+x}{1-x} \right)^{b\omega_2} & \text{for } \omega_2 > 0 \\ 1 & \text{for } \omega_2 \leq 0 \end{cases}, \end{aligned} \quad (\text{A24})$$

where

$$C'_\beta := \frac{2^{-\frac{1}{2} + \text{Re}(\beta)} \cdot e^{5\pi \cdot |Im(\beta)|/2}}{|\Gamma(\frac{1}{2} + \beta)|} \cdot B_{\frac{1}{2} + \text{Re}(\beta)}. \quad (\text{A25})$$

Proof. First one gets from the definitions and (A22),

$$\begin{aligned} |h(\omega, \beta, x)| &= \frac{\pi^{-1} \cdot 2^{\text{Re}(\beta) - \frac{1}{2}}}{|\Gamma(\frac{1}{2} + \beta)|} \cdot (1-x)^{-b\omega_2} \cdot \\ &\quad \left| \int_{-\pi/2}^{\pi/2} e^{i \cdot (3\beta + \frac{1}{2}) \cdot t} \cdot \cos^{\beta - \frac{1}{2}}(t) \cdot (x + e^{2it})^{-(\frac{1}{2} + \beta + i b \omega)} dt \right|. \end{aligned} \quad (\text{A26})$$

For each $t \in (-\pi/2, \pi/2)$ one now has

$$x + e^{2it} = |x + e^{2it}| \cdot e^{i \cdot [t + \arctan(\frac{1-x}{1+x} \cdot \tan(t))]}, \quad (\text{A27})$$

and hence,

$$\begin{aligned}
|(x + e^{2it})^{-(\frac{1}{2} + \beta + ib\omega)}| &= |(x + e^{2it})^{b\omega_2 - \frac{1}{2} - Re(\beta)} \cdot e^{(b\omega_1 + Im(\beta)) \cdot [t + \arctan(\frac{1-x}{1+x} \cdot \tan(t))]} \\
&\leq e^{\pi \cdot |Im(\beta)|} \cdot e^{2b|\omega_1| \cdot |t|} \cdot (1-x)^{-(\frac{1}{2} + Re(\beta))} \cdot |x + e^{2it}|^{b\omega_2} \\
&\leq e^{\pi \cdot |Im(\beta)|} \cdot e^{2b|\omega_1| \cdot |t|} \cdot (1-x)^{-(\frac{1}{2} + Re(\beta))} \cdot \begin{cases} (1+x)^{b\omega_2} & \text{for } \omega_2 > 0 \\ (1-x)^{b\omega_2} & \text{for } \omega_2 \leq 0 \end{cases}.
\end{aligned} \tag{A28}$$

From (A26), (A29) follows,

$$\begin{aligned}
|h(\omega, \beta, x)| &\leq \frac{\pi^{-1} \cdot 2^{Re(\beta) - \frac{1}{2}}}{|\Gamma(\frac{1}{2} + \beta)|} \cdot e^{5\pi \cdot |Im(\beta)|/2} \cdot \int_{-\pi/2}^{\pi/2} e^{2b|\omega_1| \cdot |t|} \cdot \cos^{Re(\beta) - \frac{1}{2}}(t) dt \cdot \\
&\quad (1-x)^{-(\frac{1}{2} + Re(\beta))} \cdot \begin{cases} \left(\frac{1+x}{1-x}\right)^{b\omega_2} & \text{for } \omega_2 > 0 \\ 1 & \text{for } \omega_2 \leq 0 \end{cases} \\
&= \frac{\pi^{-1} \cdot 2^{Re(\beta) + \frac{1}{2}}}{|\Gamma(\frac{1}{2} + \beta)|} \cdot e^{5\pi \cdot |Im(\beta)|/2} \cdot \int_0^{\pi/2} e^{-2b|\omega_1| \cdot |t|} \cdot \sin^{Re(\beta) - \frac{1}{2}}(t) dt \cdot \\
&\quad (1-x)^{-(\frac{1}{2} + Re(\beta))} \cdot \begin{cases} \left(\frac{1+x}{1-x}\right)^{b\omega_2} & \text{for } \omega_2 > 0 \\ 1 & \text{for } \omega_2 \leq 0 \end{cases},
\end{aligned} \tag{A29}$$

and from this (A24) by using (A1). ■

From (A7), (A24) and the continuity of K a straightforward calculation provides the following estimate.

Lemma 8. Let $\omega \in \mathbb{C} \setminus (q(A) \cup -q(A))$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$ be given. Then,

$$\begin{aligned}
|K(\omega, x, y)| &\leq \pi^2 b C'_\alpha C'_{-\alpha} \cdot \frac{e^{2\pi b|\omega_1|}}{|\cos(2\pi\alpha) + \cosh(2\pi b\omega)|} \cdot (1 + 4b^2\omega_1^2)^{-\frac{1}{2}} \cdot H_1(x, y) \cdot \\
&\quad \begin{cases} e^{\omega_2 \cdot D(x, y)} & \text{for } \omega_2 > 0 \\ e^{\omega_2 \cdot |x-y|} & \text{for } \omega_2 \leq 0 \end{cases},
\end{aligned} \tag{A30}$$

where,

$$H_1(x, y) := \begin{cases} (1 + e^{-2x/b})^{\frac{1}{2} + Re(\alpha)} \cdot (1 + e^{2y/b})^{\frac{1}{2} - Re(\alpha)} & \text{for } y \leq x \\ (1 + e^{-2y/b})^{\frac{1}{2} + Re(\alpha)} \cdot (1 + e^{2x/b})^{\frac{1}{2} - Re(\alpha)} & \text{for } y > x \end{cases}, \tag{A31}$$

and,

$$D(x, y) := |x - y| + b \cdot \begin{cases} \ln(1 + 2e^{-2x/b}) + \ln(1 + 2e^{2y/b}) & \text{for } y \leq x \\ \ln(1 + 2e^{-2y/b}) + \ln(1 + 2e^{2x/b}) & \text{for } y > x \end{cases}. \tag{A32}$$

Note that the functions H_1 and D are symmetric. Obviously (A24) implies (32), where,

$$C_1(g, f) := \pi^2 b C'_\alpha C'_{-\alpha} \cdot \iint_{\mathbb{R}^2} g^*(x) H_1(x, y) f(y) dx dy. \tag{A33}$$

In the following an estimate is given on the members of the sequence (39). The derivation of this estimate uses the following Lemmata.

Lemma 9. Let $\beta \in \mathbb{C}$ with $-1/2 < Re(\beta) < 1/2$, $z \in \mathbb{C}$ with $|z| < 1$ and $k \in \mathbb{N}$. Then the following recursion holds,

$$\begin{aligned}
&F\left(-(k+2), -(k+2) + 2\beta, \frac{1}{2} + \beta - (k+2), z\right) = \\
&(1 - 2z)F\left(-(k+1), -(k+1) + 2\beta, \frac{1}{2} + \beta - (k+1), z\right) + \\
&\frac{(k+1)(k+1 - 2\beta)}{[k+2 - (\frac{1}{2} + \beta)][k+1 - (\frac{1}{2} + \beta)]} z(1-z)F\left(-k, -k + 2\beta, \frac{1}{2} + \beta - k, z\right).
\end{aligned} \tag{A34}$$

Proof. The proof is an straightforward consequence of Formulae 15.2.2, 15.5.1 and 5.5.3 of [1]. ■

Lemma 10. Let $\beta \in \mathbb{C}$ with $-1/2 < \operatorname{Re}(\beta) < 1/2$ and $y \in [0, 1)$ Then for every $k \in \mathbb{N}$ the following estimate holds,

$$\left| F\left(-k, -k + 2\beta, \frac{1}{2} + \beta - k, y\right) \right| \leq 1. \quad (\text{A35})$$

Proof. The estimate (A35) follows from Lemma 9 using induction along with the following estimate for each $k \in \mathbb{N}$,

$$\left| \frac{(k+1)(k+1-2\beta)}{\left[k+2-\left(\frac{1}{2}+\beta\right)\right]\left[k+1-\left(\frac{1}{2}+\beta\right)\right]} \right| \leq 2. \quad (\text{A36})$$

■ The following inequalities for the elements of the quasinormal mode sequence (39) are straightforward consequences of the definitions and Lemma 10 (as well as of Formulae 6.1.17, 6.1.26, 6.1.22 of [1]). For given $x \in \mathbb{R}$, $y \in \mathbb{R}$, $t \in \mathbb{R}$ and $k \in \mathbb{N}$ one gets,

$$\left| c_{\omega_k^\pm} u_{\omega_k^\pm}(y) u_{\omega_k^\pm}(x) e^{i\omega_k^\pm t} \right| \leq a_k^\pm \left(e^{-(t-D_s(x,y))/b} \right)^{\frac{1}{2} \pm \operatorname{Re}(\alpha) + k}, \quad (\text{A37})$$

where

$$a_k^\pm := \frac{1}{4\pi} |\cot(\pi\alpha)| \frac{\Gamma(1 \pm 2\operatorname{Re}(\alpha))}{|\Gamma(1 \pm 2\alpha)|} \frac{\left(\Gamma\left(\frac{1}{2} \pm \operatorname{Re}(\alpha) + k\right)\right)^2}{\Gamma(1+k)\Gamma(1 \pm 2\operatorname{Re}(\alpha) + k)}. \quad (\text{A38})$$

Further using 6.1.22 of [1] it is easy to see that there is a positive constant C_α such that

$$|a_k^\pm| \leq C_\alpha. \quad (\text{A39})$$

Hence with such a constant C_α one gets for given $x \in \mathbb{R}$, $y \in \mathbb{R}$, $t \in \mathbb{R}$ and $k \in \mathbb{N}$:

$$\left| c_{\omega_k^\pm} u_{\omega_k^\pm}(y) u_{\omega_k^\pm}(x) e^{i\omega_k^\pm t} \right| \leq C_\alpha \left(e^{-(t-D_s(x,y))/b} \right)^{\frac{1}{2} \pm \operatorname{Re}(\alpha) + k}. \quad (\text{A40})$$

For given $x \in \mathbb{R}$ and $y \in \mathbb{R}$ from the last estimate follows the absolute and uniform summability of

$$\left(c_\omega u_\omega(y) u_\omega(x) e^{i\omega t} \right)_{\omega \in q(A)}, \quad (\text{A41})$$

on every compact subset of $(D_s(x, y), \infty) \times \mathbb{R}$ and hence also the analyticity of the function which associates to each $t \in (D_s(x, y), \infty) \times \mathbb{R}$ the value

$$\sum_{\omega \in q(A)} c_\omega u_\omega(y) u_\omega(x) e^{i\omega t}. \quad (\text{A42})$$

A further consequence of the estimate is that

$$\left(c_\omega \cdot \int_{-\infty}^{+\infty} u_\omega(y) f(y) dy \cdot \int_{-\infty}^{+\infty} g^*(x) u_\omega(x) dx \cdot e^{i\omega t} \right)_{\omega \in q(A)}, \quad (\text{A43})$$

is *absolutely and uniformly summable* on $[t_0, \infty) \times K_0$, where $t_0 > M_s(g, f)$, K_0 is any compact subset of \mathbb{R} and

$$M_s(g, f) := \max\{D_s(x, y) : x \in \operatorname{supp}(g) \text{ and } y \in \operatorname{supp}(f)\}. \quad (\text{A44})$$

Hence also the analyticity of the function which associates to each $t \in (M_s(x, y), \infty) \times \mathbb{R}$ the value

$$\sum_{\omega \in q(A)} c_\omega \cdot \int_{-\infty}^{+\infty} u_\omega(y) f(y) dy \cdot \int_{-\infty}^{+\infty} g^*(x) u_\omega(x) dx \cdot e^{i\omega t}. \quad (\text{A45})$$

The remainder of this appendix considers the sequence

$$\left(c_\omega [u_\omega(0)]^2 \right)_{\omega \in q(A)}, \quad (\text{A46})$$

which is a special case of (A41) for $x = y = 0$ and $t = 0$. This is interesting because for this case $t < D_s(x, y)$, which was not considered up to now. In the following it will be shown that *this sequence is not absolutely summable*.

First, after some computation, which uses Formulae 15.4.19, 8.6.1, 6.1.17, 6.1.18, of [1], it can be seen that

$$u_{\omega_{2k+1}^-}(0) = 0, \quad (\text{A47})$$

and that

$$\left| c_{\omega_{2k}^-} \left[u_{\omega_{2k}^-}(0) \right]^2 \right| = \frac{1}{2\pi} |\cot(\pi\alpha)| \frac{\Gamma(k + \frac{1}{2})\Gamma(k - \alpha + \frac{1}{2})}{\Gamma(k+1)\Gamma(k - \alpha + 1)}. \quad (\text{A48})$$

both for each $k \in \mathbb{N}$. Further, using Formulae 6.1.17, 6.1.26, 6.2.1 of [1] (as well as Fubini's theorem and Tonelli's theorem) one gets for each $n \in \mathbb{N}$,

$$\begin{aligned} \sum_{k=0}^n \left| c_{\omega_{2k}^-} \left[u_{\omega_{2k}^-}(0) \right]^2 \right| &\geq \frac{1}{2\pi} \frac{|\cos(\pi Re(\alpha))|}{|\sin(\pi\alpha)|} \sum_{k=0}^n \frac{\Gamma(k + \frac{1}{2})\Gamma(k - Re(\alpha) + \frac{1}{2})}{\Gamma(k+1)\Gamma(k - Re(\alpha) + 1)} \\ &= \frac{1}{2\pi^2} \frac{|\cos(\pi Re(\alpha))|}{|\sin(\pi\alpha)|} \iint_{(0,1)^2} \frac{1 - (ts)^{n+1}}{1 - ts} [ts(1-t)(1-s)]^{-1/2} s^{-Re(\alpha)} dt ds. \end{aligned} \quad (\text{A49})$$

The proof that (A46) is not absolutely summable proceeds indirectly. From the assumption that it is absolutely summable it follows by (A49) and the monotonous convergence theorem that the function defined by

$$(1 - ts)^{-1} [ts(1-t)(1-s)]^{-1/2} s^{-Re(\alpha)}, \quad (\text{A50})$$

for each $t \in (0, 1)$ and $s \in (0, 1)$ is integrable on $(0, 1)^2$. Hence using the substitution

$$t = \sin^2(\tau) \quad , \quad \tau \in (0, \pi/2), \quad (\text{A51})$$

and Fubini's theorem it follows that the function defined by

$$s^{-(\frac{1}{2} + Re(\alpha))} (1 - s)^{-1}, \quad (\text{A52})$$

for each $s \in (0, 1)$ is integrable on $(0, 1)$, which is false. Hence (A46) is not absolutely summable.

Using similar methods it can be shown that the sequence

$$\sum_{k=0}^n \left(c_{\omega_k^-} \left[u_{\omega_k^-}(0) \right]^2 + c_{\omega_k^+} \left[u_{\omega_k^+}(0) \right]^2 \right), \quad (\text{A53})$$

converges for $n \rightarrow \infty$. The proof of this is not given here. Note that this does not contradict the fact that (A46) is not absolutely summable. The “sum” of (A46) just *depends on the order of the summation*.

APPENDIX B:

This appendix gives a (not completely rigorous) derivation for the representation (16) used in this paper for the solution to the initial value problem. The derivation uses the “Laplace method” of [11]. To aid further applications of this general method, the derivation is provided for a more general wave equation (B1) than used in this paper.

Take as given J , V , and f , where J is a non empty (bounded or unbounded) interval of \mathbb{R} , V is a continuous real-valued function on J , and f is a square integrable complex-valued function on J . Let ϕ_f be a given complex-valued function on $\mathbb{R} \times J$, which is two times continuously partially differentiable and which satisfies

$$\frac{\partial^2 \phi_f}{\partial t^2}(t, x) - \frac{\partial^2 \phi_f}{\partial x^2}(t, x) + V(x)\phi_f(t, x) = 0, \quad (\text{B1})$$

for each $t \in \mathbb{R}$ and $x \in J$. In addition ϕ_f satisfies the initial conditions

$$\phi_f(0, x) = 0, \quad \frac{\partial \phi_f}{\partial t}(0, x) = f(x), \quad (\text{B2})$$

for each $t \in \mathbb{R}$ and $x \in J$. Finally, let ϵ be a given strictly positive real (otherwise arbitrary) number.

By Laplace transforming (B1) and using (B2) one gets the representations (B11), (B12) of ϕ_f below as follows. Defining

$$\psi_f(t, x) := e^{-\epsilon t} \phi_f(t, x), \quad (\text{B3})$$

for each $t \in \mathbb{R}$, $x \in J$ and assuming the boundedness of

$$\psi_f(\cdot, x), \quad \frac{\partial \psi_f}{\partial t}(\cdot, x), \quad \frac{\partial^2 \psi_f}{\partial t^2}(\cdot, x), \quad (\text{B4})$$

on each $[0, \infty)$ for each $x \in J$ one gets from (B1), (B2) for arbitrary $x \in J$ and $s \in (0, \infty) \times \mathbb{R}$,

$$\int_0^\infty e^{-st} \cdot \left(-\frac{\partial^2 \psi_f}{\partial x^2}(t, x) + [V(x) + (s + \epsilon)^2] \cdot \psi_f(t, x) \right) dt = f(x). \quad (\text{B5})$$

From this, assuming the uniformly boundedness of

$$\psi_f(\cdot, y), \quad \frac{\partial \psi_f}{\partial t}(\cdot, y), \quad \frac{\partial^2 \psi_f}{\partial t^2}(\cdot, y), \quad (\text{B6})$$

for y from a neighbourhood of x , one concludes

$$-(\Psi_f(s, \cdot))''(x) + [V(x) - (-is - i\epsilon)^2] \Psi_f(s, x) = f(x), \quad (\text{B7})$$

where

$$\Psi_f(s, y) := \int_0^\infty e^{-st} \psi_f(t, y) dt, \quad (\text{B8})$$

for each $y \in J$. Note that, roughly speaking, the $-i\epsilon$ also guarantees the unique solvability of (B7) in $L^2(\mathbb{R})$ for the limiting cases where s is purely imaginary. This fact will be used in (B8) for inverting the Laplace transform. Formal inversion of (B7) leads to

$$\begin{aligned} \Psi_f(s, x) &= G_f((\omega - i \cdot (\sigma + \epsilon))^2, x) \\ &:= \int_J G((\omega - i(\sigma + \epsilon))^2, x, y) f(y) dy, \end{aligned} \quad (\text{B9})$$

where $G(\omega - i(\sigma + \epsilon))^2, \cdot, \cdot)$ is a Green's function for the formal differential operator

$$-\frac{d^2}{dx^2} + V - [\omega - i(\sigma + \epsilon)]^2, \quad (\text{B10})$$

which one arrives at by the method of variation of constants. Here σ, ω denote the real and imaginary parts of s , respectively. Note that for the choice of an appropriate Green's function it may be necessary to impose further boundary conditions on the solutions of (B1). By assuming the square integrability of $\psi_f(\cdot, x)$ on $[0, \infty)$ the inversion of the Laplace transform in (B8) can be performed using the Fourier inversion theorem for square integrable functions on the real line. In this way one gets from (B8), (B9) the representations,

$$F^{-1} G_f((\cdot - i\epsilon)^2, x) = \sqrt{2\pi} \cdot \begin{cases} \psi_f(t, x) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}, \quad (\text{B11})$$

where F denotes the unitary linear Fourier transformation on $L^2(\mathbb{R})$ (defined according to [10], Volume II) as well as for (Lebesgue-) almost all $t \in [0, \infty)$,

$$\phi_f(t, x) = \frac{1}{2\pi} \lim_{\nu \rightarrow \infty} \int_{-\nu}^{\nu} e^{it \cdot (\omega - i\epsilon)} G_f((\omega - i\epsilon)^2, x) d\omega. \quad (\text{B12})$$

Note that the limit in the last formula is essential since from the assumptions made one can only conclude that the integrand is square integrable (but not integrable) over \mathbb{R} . Moreover note that the right hand side of (B9) and the left

hand side of (B12) are independent of ϵ , which reflects the fact that in the inversion of the Laplace transformation there is some freedom in the choice of contour. Starting from (B9) one can arrive at (16) in the following way. Let g be an arbitrarily chosen infinitely often differentiable complex-valued function on \mathbb{R} having compact support. Assuming the uniform boundedness of ψ_f on $\mathbb{R} \times \text{supp}(g)$ one gets from (B8), (B9),

$$\Psi_{g,f}(s, x) = G_{g,f}((\omega - i \cdot \epsilon)^2), \quad (\text{B13})$$

where

$$\begin{aligned} \Psi_{g,f}(s) &:= \int_0^\infty e^{-st} \cdot \langle g|\psi_f(t, \cdot) \rangle dt, \\ \langle g|\psi_f(t, \cdot) \rangle &:= \int_J g^*(x) \psi_f(t, x) dx \quad \text{for each } t \in \mathbb{R}, \\ G_{g,f}((\omega - i \cdot \epsilon)^2) &:= \int_J g^*(x) G_f((\omega - i \cdot \epsilon)^2, x) dx. \end{aligned} \quad (\text{B14})$$

Assuming the square integrability of the function which associates to each $t \in [0, \infty)$ the value of $\langle g|\psi_f(t, \cdot) \rangle$ one gets from (B13) by the Fourier inversion theorem,

$$[F^{-1}G_{g,f}((\cdot - i\epsilon)^2)](t) = \sqrt{2\pi} \cdot \begin{cases} \langle g|\psi_f(t, \cdot) \rangle & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}, \quad (\text{B15})$$

as well as for almost all $t \in [0, \infty)$,

$$\langle g|\psi_f(t, \cdot) \rangle = \frac{1}{2\pi} \lim_{\nu \rightarrow \infty} \int_{-\nu}^{\nu} e^{it \cdot (\omega - i\epsilon)} G_{g,f}((\omega - i\epsilon)^2) d\omega. \quad (\text{B16})$$

Formula (B16) is easily seen to be a consequence of (B15). In the following, sufficient conditions are given for the validness of the Formulae (B15) and (B16) leading to the formulae (B21) and (B22), respectively. For the terminology used in the following theorem consult for example Volume I of [10].

Theorem 1. Let X be a non trivial complex Hilbert space with the scalar product $\langle | \rangle$. Let $A : D(A) \rightarrow X$ be a densely defined, linear, self-adjoint, semibounded operator in X with spectrum $\sigma(A)$ and resolvent R , where the latter is defined by $R(\lambda) := (A - \lambda)^{-1}$ for each $\lambda \in \mathbb{C} \setminus \sigma(A)$. Define

$$\alpha := \begin{cases} 0 & \text{for } \min \sigma(A) \geq 0 \\ \sqrt{-\min \sigma(A)} & \text{for } \min \sigma(A) < 0 \end{cases}. \quad (\text{B17})$$

Also for each $\xi, \eta \in X$ define the analytic function $\mathcal{R}_{\xi, \eta} : \mathbb{R} \times (-\infty, -\alpha) \rightarrow \mathbb{C}$ by

$$\mathcal{R}_{\xi, \eta}(\omega + i\sigma) := \langle \xi | R((\omega + i\sigma)^2) \eta \rangle, \quad (\text{B18})$$

for each $\omega \in \mathbb{R}$ and $\sigma \in (-\infty, -\alpha)$. Finally, let ξ and η be arbitrary elements of $D(A)$ and X , respectively and let ϕ_ξ be the unique element of $C^2(\mathbb{R}, X)$ satisfying for each $t \in \mathbb{R}$

$$\phi_\xi''(t) = -A\phi_\xi(t), \quad (\text{B19})$$

and the initial conditions

$$\phi_\xi(0) = 0, \quad \phi_\xi'(0) = \xi. \quad (\text{B20})$$

Then for each $\epsilon \in (\alpha, \infty)$ and almost all (in the Lebesgue sense) $t \in [0, \infty)$,

$$\langle \eta | \phi_\xi(t) \rangle = \frac{e^{\epsilon t}}{\sqrt{2\pi}} [F^{-1} \mathcal{R}_{\eta, \xi}(\cdot - i\epsilon)](t), \quad (\text{B21})$$

and

$$\langle \eta | \phi_\xi(t) \rangle = \frac{1}{2\pi} \lim_{\nu \rightarrow \infty} \int_{-\nu}^{\nu} e^{it \cdot (\omega - i\epsilon)} \mathcal{R}_{\eta, \xi}(\omega - i\epsilon) d\omega. \quad (\text{B22})$$

This theorem is mainly a consequence of Theorem 11.6.1 in [6] and the proposition on Page 295 in Volume II of [10], and will not be proved here.

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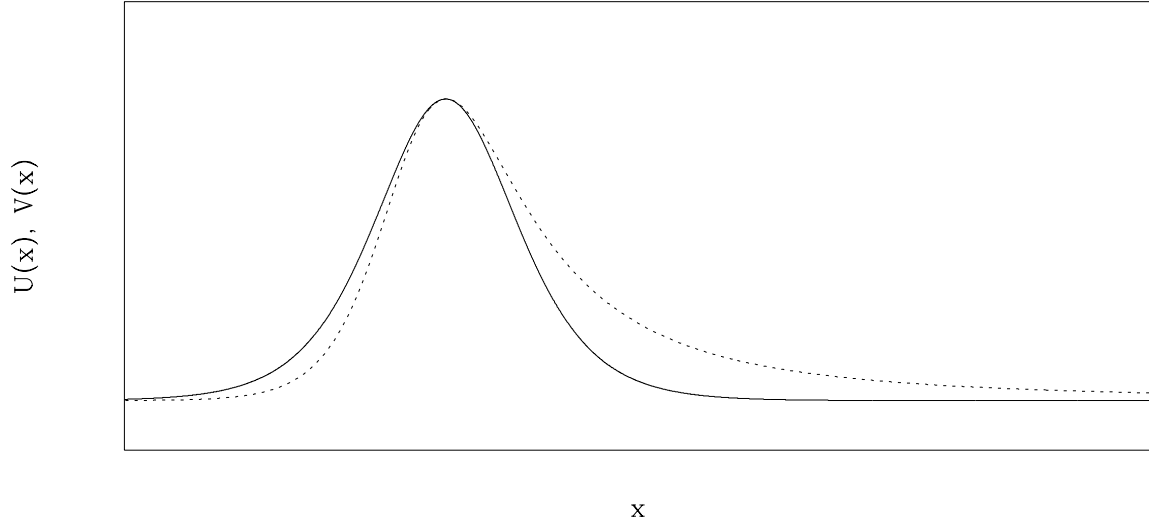


FIG. 1. A comparison of the Pöschl-Teller and Schwarzschild potentials. The parameters for the Pöschl-Teller potential are fixed by setting the maximum amplitude and the second derivative at this maximum amplitude to be equal to those for the Schwarzschild potential (with $l = 2$). That is, $V_0 = 0.61$ and $b = 2.75$. In the figure, the solid line shows the Pöschl-Teller, and the dotted line the Schwarzschild potential.

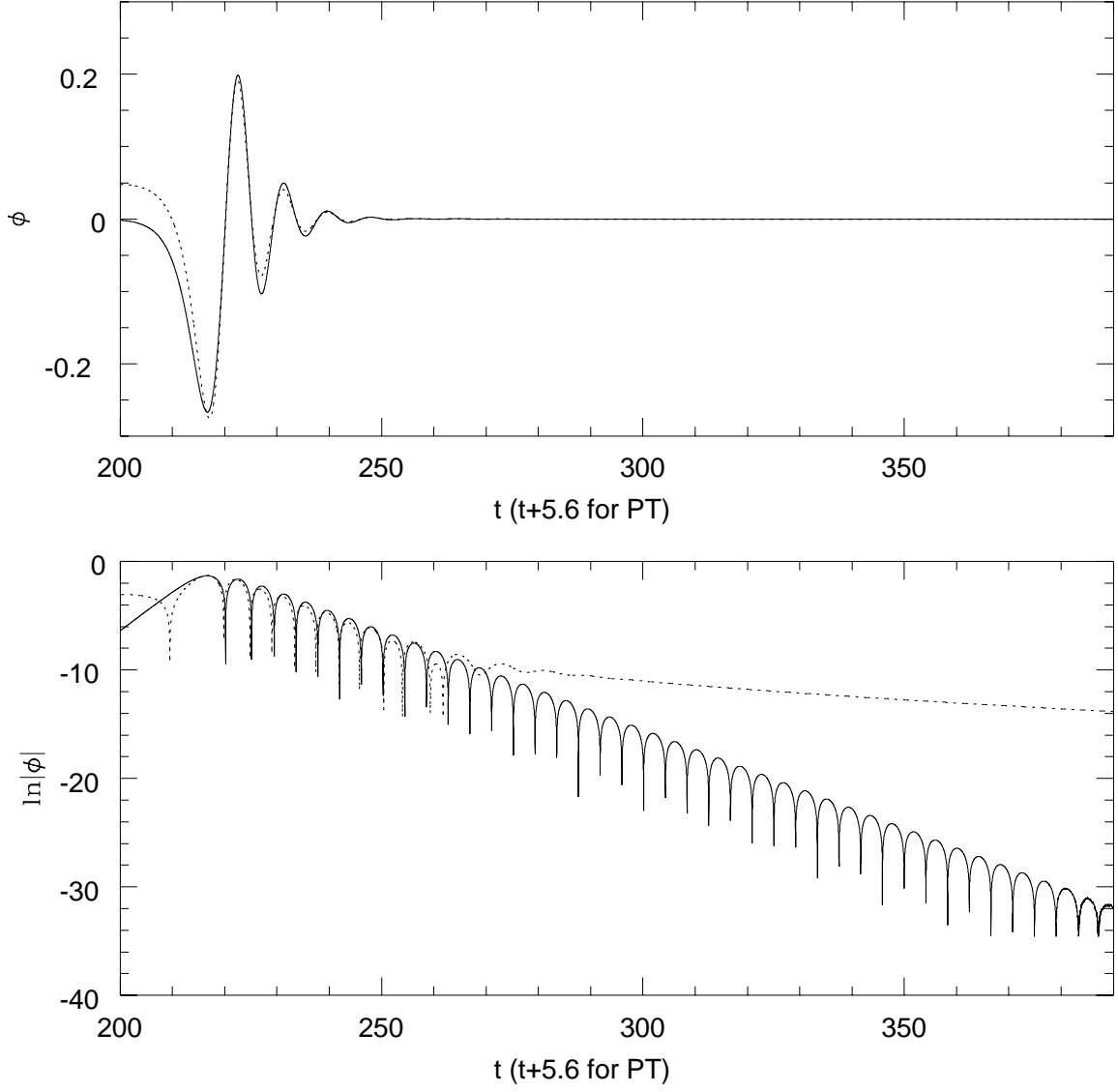


FIG. 2. Comparison of the solutions to Equation (2) (with $l = 2$) and Equation (4) (with $V_0 = 0.61$ and $b = 2.75$) from the same initial data ($\phi(0, x) = \exp(-1.5(x - 120)^2)$, $\phi_{,t}(0, x) = \phi_{,x}(0, x)$.)